Coupled higher-order nonlinear Schrödinger equations in nonlinear optics: Painlevé analysis and integrability

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A set of coupled higher-order nonlinear Schrödinger equations which can be derived from the electromagnetic pulse propagations in coupled optical waveguides and in a weakly relativistic plasma with nonlinear coupling of two polarized transverse waves is proposed. Using the Painlevé singularity structure analysis, we show that it admits the Painlevé property and hence we expect that it will exhibit soliton-type lossless propagations.

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INTRODUCTION

Electromagnetic wave propagation through nonlinear media has attracted the attention of nonlinear physicists in recent years. In particular, propagation in the form of localized pulses, i.e., solitons, through the fiber medium has important applications in communication systems [1–3]. The reason for this is due to the fact that these pulses, propagating without distortion or spreading for long distances, called optical solitons, observed experimentally since 1980, offer the possibility of achieving very high data transmission rates in optical fiber communication systems. They are considered as the future optical bits. These prospects motivate important research efforts towards the development of nonlinear aspects in optics. For example, slowly varying electromagnetic waves in a nonlinear medium are described by the nonlinear Schrödinger (NLS) equation [3–5] which also arises in various physical systems such as water waves, plasma physics, solid state physics, and so on. A further interesting fact regarding the nature of optical solitons in a fiber is the observation of higher-order effects which cannot be described by the NLS equation. An example of this higher-order effect is the Raman process which exists within the spectrum of a soliton. Recently, Kodama and Hasegawa [6] introduced a generalized higher-order NLS (HNLS) equation which takes care of the higher-order effects. The equation is of the form

$$iE_t+(\tfrac{1}{2})E_{xx}+|E|^2E+i\epsilon\{\alpha_1E_{xxx}+\alpha_2|E|^2E_x+\alpha_3E(|E|^2)_x\}=0\ , \tag{1}$$

where α_1, α_2, and α_3 represent the linear dispersion coefficient, the Kerr coefficient, and the coefficient of Raman scattering, respectively. Note that the independent variables and the parameters in Eq. (1) are different from the actual notations. In general, Eq. (1) is not completely integrable. However, if some restrictions are imposed on the parameters, one can obtain several integrable soliton possessing NLS type equations: (i) $\epsilon=0$, NLS; (ii) $\alpha_1{:}\alpha_2{:}\alpha_3=0{:}1{:}1$, derivative NLS; (iii) $\alpha_1{:}\alpha_2{:}\alpha_3=0{:}1{:}0$, derivative mixed NLS; and (iv) the Hirota equation ($\alpha_1{:}\alpha_2{:}\alpha_3=1{:}6{:}0$) and the Sasa-Satsuma case [7,8] ($\alpha_1{:}\alpha_2{:}\alpha_3=1{:}6{:}3$). Recently, using the singularity structure analysis, it has been shown that Eq. (1) is completely integrable only for the choices $\alpha_1=1$, $\alpha_2=6$, $\alpha_3=3$ [8].

The main objective of the present work is to generalize the above equation. There are several ways to generalize Eq. (1) to a set of coupled evolution equations, depending on the physical situation being modeled [1–4]. One such possibility in nonlinear optics is the following.

Let us represent $\mathbf{E}$ as a sum of right- and left-handed polarized waves. $\mathbf{E}=q_1\mathbf{e}_R+q_2\mathbf{e}_L$, where $\mathbf{e}_R$ and $\mathbf{e}_L$ are the complex unit vectors corresponding to right-handed and left-handed polarizations [3]. Using the orthogonality of $\mathbf{e}_R$ and $\mathbf{e}_L$, the general form of the coupled HNLS equation is given by

$$\begin{aligned}&iq_{1t}+(\tfrac{1}{2})q_{1xx}+(|q_1|^2+|q_2|^2)q_1+i\epsilon\{\alpha_1q_{1xxx}+\alpha_2(|q_1|^2+|q_2|^2)q_{1x}+\alpha_3q_1(|q_1|^2+|q_2|^2)_x\}=0\ ,\\&iq_{2t}+(\tfrac{1}{2})q_{2xx}+(|q_1|^2+|q_2|^2)q_2+i\epsilon\{\alpha_1q_{2xxx}+\alpha_2(|q_1|^2+|q_2|^2)q_{2x}+\alpha_3q_2(|q_1|^2+|q_2|^2)_x\}=0\ .\end{aligned} \tag{2}$$

The other physical applications of the above model will be published elsewhere.

Recently, the coupled NLS equations have been the focus of intense attention. Due to the complexity of the nature of the problem, only a little progress has been made in understanding the dynamics of the coupled nonlinear evolution equations. Some of the well studied coupled nonlinear Schrödinger equations in nonlinear optics are the following:

(1) The coupled NLS equations [1–3,9–14],

$$\begin{aligned}&iq_{1t}+(\tfrac{1}{2})q_{1xx}+(|q_1|^2+|q_2|^2)q_1=0\ ,\\&iq_{2t}+(\tfrac{1}{2})q_{2xx}+(|q_1|^2+|q_2|^2)q_2=0\ .\end{aligned} \tag{3}$$

(2) The coupled Hirota equations [17,18]

$$iq_{1t}+(\tfrac{1}{2})q_{1xx}+(|q_1|^2+|q_2|^2)q_1+i\epsilon\{q_{1xxx}+3(|q_1|^2+|q_2|^2)q_{1x}+3(q_1^*q_{1x}+q_2^*q_{2x})q_1\}=0\ ,$$
$$iq_{2t}+(\tfrac{1}{2})q_{2xx}+(|q_1|^2+|q_2|^2)q_2+i\epsilon\{q_{2xxx}+3(|q_1|^2+|q_2|^2)q_{2x}+3(q_1^*q_{1x}+q_2^*q_{2x})q_2\}=0\ , \quad (4)$$

which, in the limit $\epsilon\to 0$, is the case (1) [Eq. (3)] discussed above. Manakov [3] found the explicit soliton solutions to equations (3) with equal self and cross phase modulations. Solitary waves in the coupled NLS equations, including the nonintegrable case, are also of interest [10–16]. The Painlevé analysis of Eqs. (3) was carried out in Ref. [9]. Recently, the soliton solutions of Eqs. (4) were discussed in detail through the inverse scattering method [17]. The main aim of this paper is to establish the Painlevé property and other integrability aspects of Eq. (2). Along the lines of Refs. [7,8] we choose $\alpha_1=1$, $\alpha_2=6$, and $\alpha_3=3$. For this choice of parameters, Eq. (2) becomes

$$iq_{1t}+(\tfrac{1}{2})q_{1xx}+(|q_1|^2+|q_2|^2)q_1+i\epsilon\{q_{1xxx}+6(|q_1|^2+|q_2|^2)q_{1x}+3q_1(|q_1|^2+|q_2|^2)_x\}=0\ ,$$
$$iq_{2t}+(\tfrac{1}{2})q_{2xx}+(|q_1|^2+|q_2|^2)q_2+i\epsilon\{q_{2xxx}+6(|q_1|^2+|q_2|^2)q_{2x}+3q_2(|q_1|^2+|q_2|^2)_x\}=0\ . \quad (5)$$

PAINLEVÉ ANALYSIS

As the nature of the dynamics of Eqs. (5) are not known, we first apply the Painlevé (P) analysis to Eqs. (5) to identify the Painlevé property which then leads to the integrability properties. The P analysis is one of the systematic methods to identify the integrability cases of the nonlinear partial differential equations [19–24], i.e., the solutions which are free from movable critical manifolds.

The method for applying the Painlevé test introduced by Weiss, Tabor, and Carnevale [20] with simplifications due to Kruskal [21] involves seeking a solution of a given partial differential equation in the form

$$q(x,t)=\phi^p\sum_{j=0}^{\infty}q_j(t)\phi^j(x,t),\quad q_0\neq 0$$

with

$$\phi(x,t)=x+\psi(t)=0\ , \quad (6)$$

where $\psi(t)$ is an arbitrary analytic function of t, and $q_j(t)$, $j=0,1,2,\ldots,$ is an analytic function of t, in the neighborhood of a noncharacteristic movable singularity manifold defined by $\phi=0$.

In addition to providing a valuable first test for whether a given partial differential equation is completely integrable, other important informations relating to completely integrable equations can also be obtained from the Painlevé analysis, including the Bäcklund transformation, Lax pair, Hirota's bilinear representation, special and rational solutions, etc. [20–22]. Many of these results are obtained by truncating the Laurent series at a constant level term [23,24]. At this juncture, we would like to point out that the above integrability properties have been constructed only for uncoupled nonlinear partial differential equations. The construction of the Lax pair is still an open question for the coupled evolution equation.

In order to investigate the integrability properties of Eqs. (5), we rewrite it in terms of four complex functions a, b, c, and d by defining $q_1=a$, $q_1^*=b$, $q_2=c$, $q_2^*=d$. Consequently, we have the following equations:

$$ia_t+(\tfrac{1}{2})a_{xx}+(ab+cd)a+i\epsilon\{a_{xxx}+6(ab+cd)a_x+3(a_xb+b_xa+c_xd+cd_x)a\}=0\ ,$$
$$-ib_t+(\tfrac{1}{2})b_{xx}+(ab+cd)b-i\epsilon\{b_{xxx}+6(ab+cd)b_x+3(a_xb+b_xa+c_xd+cd_x)b\}=0\ ,$$
$$(7)$$
$$ic_t+(\tfrac{1}{2})c_{xx}+(ab+cd)c+i\epsilon\{c_{xxx}+6(ab+cd)c_x+3(a_xb+b_xa+c_xd+cd_x)c\}=0\ ,$$
$$-id_t+(\tfrac{1}{2})d_{xx}+(ab+cd)d-i\epsilon\{d_{xxx}+6(ab+cd)d_x+3(a_xb+b_xa+c_xd+cd_x)d\}=0\ .$$

Looking at the leading order behavior, we substitute $a\simeq a_0\phi^p$, $b\simeq b_0\phi^q$, $c\simeq c_0\phi^r$, $d\simeq d_0\phi^s$ in Eqs. (7) and balancing the different terms, we obtain the following results:

$$p=q=r=s=-1\ ,$$
$$a_0b_0+c_0d_0=-\tfrac{1}{2}\ . \quad (8)$$

For finding the powers at which the arbitrary functions can enter into the series, we substitute the expressions,

$$\begin{aligned} a&=a_0\phi^{-1}+a_j\phi^{j-1}\,,\\ b&=b_0\phi^{-1}+b_j\phi^{j-1}\,,\\ c&=c_0\phi^{-1}+c_j\phi^{j-1}\,,\\ d&=d_0\phi^{-1}+d_j\phi^{j-1}\,, \end{aligned} \tag{9}$$

into (7) and, keeping the leading order terms alone, we obtain the determinant in the form

$$\begin{vmatrix} \begin{matrix}(j-1)(j-2)(j-3)\\+(9j-24)a_0b_0\\+(6j-12)c_0d_0\end{matrix} & (3j-12)a_0^2 & (3j-12)a_0b_0 & (3j-12)a_0c_0 \\ (3j-12)b_0^2 & \begin{matrix}(j-1)(j-2)(j-3)\\+(9j-24)a_0b_0\\+(6j-12)c_0d_0\end{matrix} & (3j-12)b_0d_0 & (3j-12)b_0c_0 \\ (3j-12)b_0c_0 & (3j-12)a_0c_0 & \begin{matrix}(j-1)(j-2)(j-3)\\+(9j-24)c_0d_0\\+(6j-12)a_0b_0\end{matrix} & (3j-12)c_0^2 \\ (3j-12)b_0d_0 & (3j-12)a_0d_0 & (3j-12)d_0^2 & \begin{matrix}(j-1)(j-2)(j-3)\\+(9j-24)c_0d_0\\+(6j-12)a_0b_0\end{matrix} \end{vmatrix}=0\,. \tag{10}$$

(11)

Using (8) in (10) and solving the determinant, we obtain

$$j^{12}-24j^{11}+245j^{10}-1374j^9+4524j^8-8424j^7+6656j^6+3840j^5-11264j^4+6144j^3=0\,.$$

Solving Eq. (11), the resonances are found to be

$$j=-1,0,0,0,2,2,2,3,4,4,4,4\,. \tag{12}$$

For each resonance value, there is a compatibility condition which must be identically satisfied so that Eqs. (5) have a general solution of the form (9). The resonances $j=-1$ and $j=0,0,0$ correspond to the fact that $\psi(t)$ is arbitrary, and that there is only one equation defining a_0, b_0, c_0, and d_0 (so three, say, b_0, c_0, and d_0 are arbitrary), respectively.

Proceeding further and equating the coefficients of $(\phi^{-3},\phi^{-3},\phi^{-3},\phi^{-3})$, we obtain

$$\begin{bmatrix} 3a_0b_0-1 & 3a_0^2 & 3a_0d_0 & 3a_0c_0\\ 3b_0^2 & 3a_0b_0-1 & 3b_0d_0 & 3b_0c_0\\ 3b_0c_0 & 3a_0c_0 & 3a_0b_0-1 & 3c_0^2\\ 3b_0d_0 & 3a_0d_0 & 3d_0^2 & 3a_0b_0-1 \end{bmatrix} \begin{bmatrix} a_1\\ b_1\\ c_1\\ d_1\end{bmatrix} = \frac{-1}{6i\epsilon}\begin{bmatrix} -a_0\\ b_0\\ -c_0\\ d_0\end{bmatrix}\,. \tag{13}$$

From (13), we get

$$\begin{aligned} a_1&=-a_0/6i\epsilon\,,\\ b_1&=b_0/6i\epsilon\,,\\ c_1&=-c_0/6i\epsilon\,,\\ d_1&=d_0/6i\epsilon\,. \end{aligned} \tag{14}$$

On the other hand, the coefficients of $(\phi^{-2},\phi^{-2},\phi^{-2},\phi^{-2})$ in Eqs. (5) reduce to a single equation

$$b_0a_2+a_0b_2+d_0c_2+c_0d_2=\frac{-1}{6i\epsilon}\psi_t \tag{15}$$

so that three of the four functions a_2, b_2, c_2, and d_2 are arbitrary which corresponds to $j=2,2,2$.

Similarly from the powers of $(\phi^{-1},\phi^{-1},\phi^{-1},\phi^{-1})$ and $(\phi^0,\phi^0,\phi^0,\phi^0)$, we find that Eqs. (5) admit the sufficient number of arbitrary functions and hence Eqs. (7) possess the Painlevé property and hence they are expected to be integrable.

Once the integrability is proved, one can construct the integrability properties by truncating the Laurent series at a constant level term. Though it is possible to construct the integrability properties such as the Bäcklund transformations, (BT) and bilinear forms from the P analysis, the construction of the Lax pair for the coupled NLS type equation is still an open question. For the construction of the auto BT, we truncate the series (9) up to a constant level term, i.e., a_j, b_j, c_j, and d_j are equal to zero for $j \geq 2$. Thus, the associated BT leads to

$$\begin{aligned} &a = a_0\phi^{-1} + a_1, \quad b = b_0\phi^{-1} + b_1 , \\ &c = c_0\phi^{-1} + c_1, \quad d = d_0\phi^{-1} + d_1 , \end{aligned} \tag{16}$$

where (a,a_1), (b,b_1), (c,c_1), and (d,d_1) satisfy Eqs. (7). So, Eqs. (16) may be treated as a auto BT of Eqs. (7). As the construction of the Lax pair from the truncation of a Laurent series is very difficult for the coupled NLS type equations, for further discussion we restrict to the construction of a bilinear form for Eqs. (7). For this, we assume $a_1 = b_1 = c_1 = d_1 = 0$ and using (16) in (7), we obtain the following set of equations

$$\begin{aligned}
&\frac{i}{\phi^2}D_t(a_0\cdot\phi) + \frac{1}{2\phi^2}D_x^2(a_0\cdot\phi) + \frac{a_0}{\phi^3}[\Gamma + D_x^2(\phi\cdot\phi)] + \frac{i\varepsilon}{\phi^2}D_x^3(a_0\cdot\phi) \\
&\qquad - \frac{6i\varepsilon}{\phi^4}D_x(a_0\cdot\phi)D_x^2(\phi\cdot\phi) - \frac{3i\varepsilon}{\phi^4}[\phi_x^2 D_x(a_0\cdot\phi) + a_0\phi_x D_x^2(\phi\cdot\phi)] = 0 , \\
&-\frac{i}{\phi^2}D_t(b_0\cdot\phi) + \frac{1}{2\phi^2}D_x^2(b_0\cdot\phi) + \frac{b_0}{\phi^3}[\Gamma + D_x^2(\phi\cdot\phi)] - \frac{i\varepsilon}{\phi^2}D_x^3(b_0\cdot\phi) \\
&\qquad + \frac{6i\varepsilon}{\phi^4}D_x(b_0\cdot\phi)D_x^2(\phi\cdot\phi) + \frac{3i\varepsilon}{\phi^4}[\phi_x^2 D_x(b_0\cdot\phi) + b_0\phi_x D_x^2(\phi\cdot\phi)] = 0 , \\
&\frac{i}{\phi^2}D_t(c_0\cdot\phi) + \frac{1}{2\phi^2}D_x^2(c_0\cdot\phi) + \frac{c_0}{\phi^3}[\Gamma + D_x^2(\phi\cdot\phi)] + \frac{i\varepsilon}{\phi^2}D_x^3(c_0\cdot\phi) \\
&\qquad - \frac{6i\varepsilon}{\phi^4}D_x(c_0\cdot\phi)D_x^2(\phi\cdot\phi) - \frac{3i\varepsilon}{\phi^4}[\phi_x^2 D_x(c_0\cdot\phi) + c_0\phi_x D_x^2(\phi\cdot\phi)] = 0 , \\
&-\frac{i}{\phi^2}D_t(d_0\cdot\phi) + \frac{1}{2\phi^2}D_x^2(d_0\cdot\phi) + \frac{d_0}{\phi^3}[\Gamma + D_x^2(\phi\cdot\phi)] - \frac{i\varepsilon}{\phi^2}D_x^3(d_0\cdot\phi) \\
&\qquad + \frac{6i\varepsilon}{\phi^4}D_x(d_0\cdot\phi)D_x^2(\phi\cdot\phi) + \frac{3i\varepsilon}{\phi^4}[\phi_x^2 D_x(d_0\cdot\phi) + d_0\phi_x D_x^2(\phi\cdot\phi)] = 0 ,
\end{aligned} \tag{17}$$

where $\Gamma = a_0 b_0 + c_0 d_0$ and the D operators are defined by

$$D_t^n D_x^m a\cdot b = \left[\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right]^n \left[\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right]^m a(x,t)\cdot b(x',t')\bigg|_{\substack{x'=x \\ t'=t}} .$$

By equating the different powers of ϕ to zero, after some simplifications, one can get Hirota's bilinear operators and also construct the N-soliton solutions in the usual way.

CONCLUSIONS

In this paper, we have formulated a more generalized set of coupled higher-order nonlinear Schrödinger equations which can be derived from nonlinear optics by considering the electromagnetic wave E as a sum of right- and left-hand polarized waves. The same equations can also be derived from a weakly relativistic plasma with nonlinear coupling of two polarized transverse waves. Then, by choosing the parameters as in the case of the corresponding integrable uncoupled case, we applied the Painlevé singularity structure analysis and established that for this particular choice of parameters, the proposed equations possess the Painlevé property. As the system of proposed coupled equations admit the P property, we expect that they will admit the soliton type of propagations. Future papers will focus mainly on the construction of the linear eigenvalue problem and soliton solutions of Eqs. (5) and other integrability cases of the generalized equations (2).

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